Lecture 14: Singular Homology

Chain complex

Definition

Let R be a commutative ring. A chain complex over R is a sequence of R-module maps

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0 \ \forall n$. When R is not specified, we mean chain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the map by ∂ and the chain complex by $(C_\bullet, \partial)$. Then $\partial_n = \partial|_{C_n}$ and $\partial^2 = 0$.

Definition

A chain map $f\colon C_\bullet \to C'_\bullet$ between two chain complexes over R is a sequence of R-module maps $f_n : C_n \to C'_n$ such that the following diagram is commutative

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & C'_n & \xrightarrow[\partial'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}
$$

This can be simply expressed as

$$\boxed{f \circ \partial = \partial' \circ f}$$

Definition

We define the category $\underline{\mathrm{Ch}_\bullet(\mathrm{R})}$ whose objects are chain complexes over R and morphisms are chain maps. We simply write $\underline{\mathbf{Ch}}_\bullet$ when $R = \mathbb{Z}$.

Definition

Given a chain complex $(C_\bullet, \partial)$, we define its *n*-cycles Z_n and *n*-boundaries B_n by

$$Z_n = \mathrm{Ker}(\partial : C_n \to C_{n-1}), \quad B_n = \mathrm{Im}(\partial : C_{n+1} \to C_n).$$

The equation $\partial^2 = 0$ implies $B_n \subset Z_n$. We define the *n*-th homology group by

$$\boxed{H_n(C_\bullet, \partial) := \frac{Z_n}{B_n} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}}.$$

A chain complex $C_\bullet$ is called acyclic or exact if

$$H_n(C_\bullet) = 0 \quad \text{for any } n.$$

Proposition

The n-th homology group defines a functor

$$H_n : \mathbf{Ch}_\bullet \to \mathbf{Ab}.$$

Proof.

We only need to check any $f\colon C_\bullet \to C'_\bullet$ induces a group homomorphism

$$f_* : H_n(C_\bullet) \to H_n(C'_\bullet).$$

This is because

- if $\alpha \in Z_n(C_\bullet)$, then $f(\alpha) \in Z_n(C'_\bullet)$;
- if $\alpha \in B_n(C_\bullet)$, then $f(\alpha) \in B_n(C'_\bullet)$.

□

Definition

A chain map $f\colon C_\bullet \to D_\bullet$ is called a quasi-isomorphism if

$$f_* : H_n(C_\bullet) \to H_n(D_\bullet)$$

is an isomorphism for all n.

Definition

A chain homotopy $f \stackrel{s}{\simeq} g$ between two chain maps $f, g : C_\bullet \to C'_\bullet$ is a sequence of homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1} \circ \partial_n + \partial'_{n+1} \circ s_n,$$

or simply

$$\boxed{f - g = s \circ \partial + \partial' \circ s}.$$

Two complexes $C_\bullet, C'_\bullet$ are called chain homotopy equivalent if there exists chain maps $f : C_\bullet \to C'_\bullet$ and $h : C'_\bullet \to C_\bullet$ such that

$$f \circ g \simeq 1 \quad \text{and} \quad g \circ f \simeq 1.$$

Proposition

Chain homotopy defines an equivalence relation on chain maps and compatible with compositions.

In other words, chain homotopy defines an equivalence relation on $\underline{\mathbf{Ch}}_\bullet$. We define the quotient category

$$\underline{\mathrm{h}\mathbf{Ch}}_\bullet = \underline{\mathbf{Ch}}_\bullet / \simeq .$$

Chain homotopy equivalence becomes an isomorphism in $\underline{\mathrm{h}\mathbf{Ch}}_\bullet$.

Proposition

Let f, g be chain homotopic chain maps. Then they induce identical map on homology groups

$$H_n(f) = H_n(g) : H_n(C_\bullet) \to H_n(C'_\bullet).$$

In other words, the functor H_n factor through

$$H_n : \mathbf{Ch}_\bullet \to \mathrm{h}\mathbf{Ch}_\bullet \to \mathbf{Ab}.$$

Proof.

Let $f - g = s \circ \partial + \partial' \circ s$. Let $\alpha \in C_n$ represent a class $[\alpha]$ in $H_n(C_\bullet)$. Since $\partial\alpha = 0$, we have

$$(f - g)(\alpha) = (s \circ \partial + \partial' \circ s)(\alpha) = \partial' \circ \big(s(\alpha)\big) \in B_n(C'_\bullet).$$

So $[f(\alpha)] = [g(\alpha)]$. Hence $H_n(f) = H_n(g)$ on homologies. □

Singular Homology

Definition

We define the standard n-simplex

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

We let $\{v_0, \cdots, v_n\}$ denote its vertices. Here $v_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 sits at the i-th position.

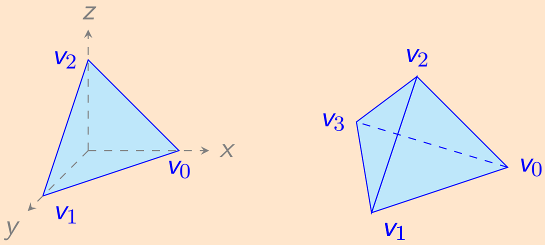


图: Standard 2-simplex Δ^2 and 3-simplex Δ^3

Definition

Let X be a topological space. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$. For each $n \geq 0$, we define $S_n(X)$ to be the free abelian group generated by all singular n-simplexes in X

$$S_n(X) = \bigoplus_{\sigma \in \mathrm{Hom}(\Delta^n, X)} \mathbb{Z}\sigma.$$

An element of $S_n(X)$ is called a singular n-chain in X.

A singular n-chain is given by a finite formal sum

$$\gamma = \sum_{\sigma \in \mathrm{Hom}(\Delta^n, X)} m_\sigma \sigma,$$

for $m_\sigma \in \mathbb{Z}$ and only finitely many m_σ's are nonzero. The abelian group structure is:

$$-\gamma := \sum_\sigma (-m_\sigma)\sigma$$

and

$$\left(\sum_\sigma m_\sigma \sigma\right) + \left(\sum_\sigma m'_\sigma \sigma\right) = \sum_\sigma (m_\sigma + m'_\sigma)\sigma.$$

Given a singular n-simplex $\sigma : \Delta^n \to X$ and $0 \leq i \leq n$, we define

$$\partial^{(i)}\sigma : \Delta^{n-1} \to X$$

to be the $(n-1)$-simplex by restricting σ to the i-th face of Δ^n whose vertices are given by $\{v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n\}$.

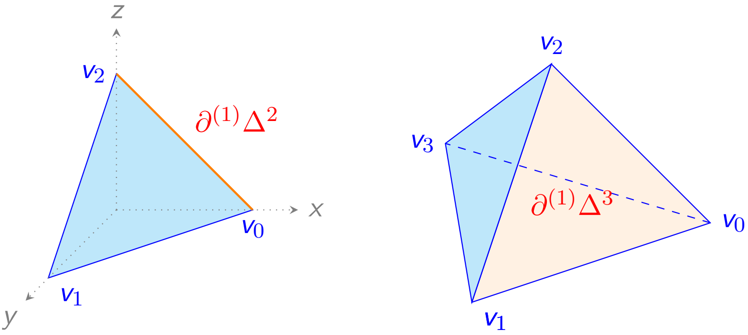


图: Faces of 2-simplex Δ^2 and 3-simplex Δ^3

Definition

We define the boundary map

$$\partial : S_n(X) \to S_{n-1}(X)$$

to be the abelian group homomorphism generated by

$$\boxed{\partial\sigma := \sum_{i=0}^{n} (-1)^i \partial^{(i)} \sigma}.$$

Given a subset $\{v_{i_1}, \cdots, v_{i_k}\}$ of the vertices of Δ^n, we will write

$\sigma|[v_{i_1}, \cdots, v_{i_k}]$ or just $[v_{i_1}, \cdots, v_{i_k}]$ (when it is clear from the context)

for restricting σ to the face of Δ^n spanned by $\{v_{i_1}, \cdots, v_{i_k}\}$. Then the boundary map can be expressed by

$$\partial[v_0, \cdots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n].$$

Proposition

$(S_\bullet(X), \partial)$ defines a chain complex, i.e., $\partial^2 = \partial \circ \partial = 0$.

Proof.

$$
\begin{aligned}
&\partial \circ \partial [v_0, \cdots, v_n] \\
=&\partial \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n] \\
=&\sum_{i<j} (-1)^i (-1)^{j+1} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] \\
&+ \sum_{j<i} (-1)^i (-1)^j [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] \\
&= 0.
\end{aligned}
$$

□

Example

Consider a 2-simplex $\sigma : \Delta^2 \to X$. Then

$$\partial\sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

and

$$\partial^2\sigma = ([v_2] - [v_1]) - ([v_2] - [v_0]) + ([v_1] - [v_0]) = 0.$$

Definition

For each $n \geq 0$, we define the n-th singular homology group of X by

$$\boxed{H_n(X) := H_n(S_\bullet(X), \partial)}.$$

Let $f: X \to Y$ be a continuous map, which gives a chain map

$$S_\bullet(f): S_\bullet(X) \to S_\bullet(Y).$$

This defines the functor of singular chain complex

$$S_\bullet: \mathbf{Top} \to \mathbf{Ch}_\bullet\,.$$

Singular homology group can be viewed as the composition

$$\mathbf{Top} \xrightarrow{S_\bullet} \mathbf{Ch}_\bullet \xrightarrow{H_n} \mathbf{Ab}\,.$$

Proposition

Let $f, g : X \to Y$ be homotopic maps. Then

$$S_\bullet(f), S_\bullet(g) : S_\bullet(X) \to S_\bullet(Y)$$

are chain homotopic. In particular, they induce identical map

$$H_n(f) = H_n(g) : H_n(X) \to H_n(Y).$$

Proof: We only need to prove that for $i_0, i_1 : X \to X \times I$,

$$S_\bullet(i_0), S_\bullet(i_1) : S_\bullet(X) \to S_\bullet(X \times I)$$

are chain homotopic. Then their composition with the homotopy $X \times I \to Y$ gives the proposition.

Let us define a homotopy

$$s : S_n(X) \to S_{n+1}(X \times I).$$

For $\sigma : \Delta^n \to X$, we define (topologically)

$$s(\sigma) : \Delta^n \times I \stackrel{\sigma\times 1}{\to} X \times I.$$

Here we treat $\Delta^n \times I$ as a collection of $(n+1)$-simplexes as follows. Let $\{v_0, \cdots, v_n\}$ denote the vertices of Δ^n. The vertices of $\Delta^n \times I$ contain two copies $\{v_0, \cdots, v_n\}$ and $\{w_0, \cdots, w_n\}$.

Then

$$\Delta^n \times I = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots v_i, w_i, w_{i+1}, \cdots, w_n]$$

cuts $\Delta^n \times I$ into $(n+1)$-simplexes.

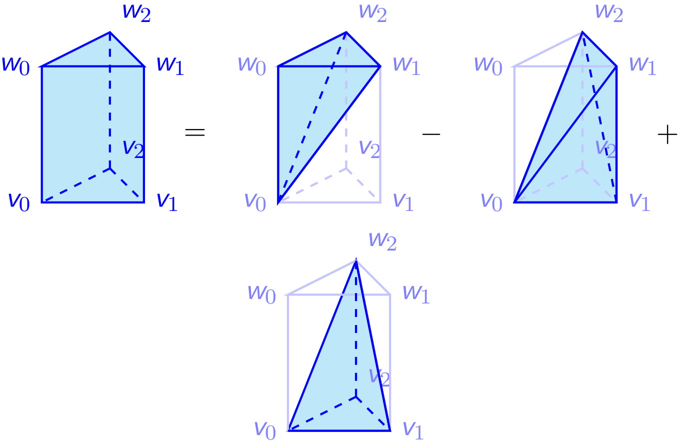


图: Decomposition of $\Delta^n \times I$ for $n = 2$

Its sum defines

$$s(\sigma) \in S_{n+1}(X \times I).$$

The following intuitive formula holds

$$\partial(\Delta^n \times I) = \Delta \times \partial I - (\partial\Delta^n) \times I$$

as singular chains. This leads to the chain homotopy

$$S_\bullet(i_1) - S_\bullet(i_0) = \partial \circ s + s \circ \partial.$$

□

Theorem

Singular homologies are homotopy invariants. They factor through

$$H_n : \mathbf{hTop} \to \mathrm{h}\mathbf{Ch}_\bullet \to \mathbf{Ab}\,.$$

Theorem (Dimension Axiom)

If X is a one-point space, then

$$H_n(X) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

Proof: For each $n \geq 0$, there is only one $\sigma_n : \Delta^n \to X$.

$$S_n(X) = \mathbb{Z} \langle \sigma_n \rangle .$$

The boundary operator is

$$\partial \langle \sigma_n \rangle = \sum_{i=0}^{n} (-1)^i \langle \sigma_{n-1} \rangle = \begin{cases} 0 & n = \text{odd} \\ \sigma_{n-1} & n = \text{even}. \end{cases}$$

The singular chain complex of X becomes

$$\cdots \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to \mathbb{Z} \to 0$$

which implies the theorem.

Some Homological algebra

The following lemma is very useful in dealing with chain complexes.

Lemma (Five Lemma)

Consider the commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Then

1. If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.
2. If f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.
3. If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism.

Definition

Let $f\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be a chain map. The mapping cone of f is the chain complex

$$cone(f)_n = C_{n-1} \oplus C'_n$$

with the differential

$$d : cone(f)_n \to cone(f)_{n-1},$$

$$d(c_{n-1}, c'_n) = (-\partial(c_{n-1}), \partial'(c'_n) - f(c_{n-1})).$$

Proposition

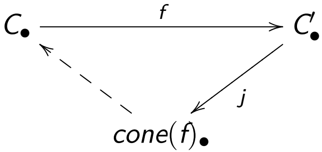
Let $f\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be a chain map.

1. There is an exact sequence

$$0 \to C'_\bullet \to cone(f)_\bullet \to C[-1]_\bullet \to 0$$

   Here $C[-1]_\bullet$ is the chain complex with $C[-1]_n := C_{n-1}$ and differential $-\partial$ where ∂ is the differential in C.
2. f is a quasi-isomorphism if and only if $cone(f)_\bullet$ is acyclic.
3. Let $j : C'_\bullet \hookrightarrow cone(f)_\bullet$ be the embedding above. Then $cone(j)_\bullet$ is chain homotopic equivalent to $C[-1]_\bullet$.

In homological algebra, a chain map f leads to a triangle



Here the dotted arrow is a chain map

$$cone(f)_\bullet \to C[-1]_\bullet.$$

This is closely related to the cofiber exact sequence. $cone(f)_\bullet$ is the analogue of homotopy cofiber of f. $C_\bullet[-1]$ is the analogue of the suspension. Then the above triangle structure can be viewed as

$$C_\bullet \xrightarrow{f} C'_\bullet \to cone(f)_\bullet \to C_\bullet[-1] \xrightarrow{f[-1]} C'_\bullet \to \cdots$$